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Probability Theory and Applications (MA208)
Problem Sheet - 1

Introduction to Probability

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1. Suppose that the universal set consists of the positive integers from 1 through 10. Let $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{5, 6, 7\}$. List the members of the following sets.

   (a) $\bar{A} \cap B$
   (b) $\bar{A} \cup B$
   (c) $\overline{\bar{A} \cap \bar{B}}$
   (d) $\overline{A} \cap (\overline{B \cap C})$
   (e) $\overline{A \cap (B \cup C)}$

2. Suppose that the universal set U is given by $U = \{x : 0 \leq x \leq 2\}$. Let the sets A and B be defined as follows: $A = \{x : \frac{1}{2} < x \leq 1\}$ and $B = \{x : \frac{1}{4} \leq x < \frac{3}{2}\}$. Describe the following sets.

   (a) $\overline{A \cup B}$
   (b) $A \cup \bar{B}$
   (c) $\overline{A \cap B}$
   (d) $\bar{A} \cap B$

3. Which of the following relationships are true?

   (a) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$
   (b) $(A \cup B) = ((A \cap \bar{B}) \cup B)$
   (c) $\bar{A} \cap B = A \cup B$
   (d) $(\overline{A \cup B}) \cap C = \bar{A} \cap \bar{B} \cap \bar{C}$
   (e) $(A \cap B) \cap (\bar{B} \cap C) = \emptyset$

4. Suppose that the universal set consists of all points (x, y) both of whose coordinates are integers and which lie inside or on the boundary of the square bounded by the lines $x = 0, y = 0, x = 6$, and $y = 6$. List the members of the following sets.

   (a) $A = \{(x, y) : x^2 + y^2 \leq 6\}$
   (b) $B = \{(x, y) : y \leq x^2\}$
   (c) $C = \{(x, y) : x \leq y^2\}$
   (d) $B \cap C$
   (e) $(B \cup A) \cap \bar{C}$

5. Use Venn diagrams to establish the following relationships.

   (a) $A \subset B$ and $B \subset C$ imply that $A \subset C$
   (b) $A \subset B$ implies that $A = A \cap B$

(c) $A \subset B$ implies that $\bar{B} \subset \bar{A}$

(d) $A \subset B$ implies that $A \cup C \subset B \cup C$

(e) $A \cap B = \emptyset$ and $C \subset A$ imply that $B \cap C = \emptyset$

6. Items coming off a production line are marked defective (D) or nondefective (N). Items are observed and their condition listed. This is continued until two consecutive defectives are produced or four items have been checked, whichever occurs first. Describe a sample space for this experiment.

7. (a) A box of N light bulbs has $r(r < N)$ bulbs with broken filaments. These bulbs are tested, one by one, until a defective bulb is found. Describe a sample space for this experiment.

   (b) Suppose that the above bulbs are tested, one by one, until all defectives have been tested. Describe the sample space for this experiment.

8. Consider four objects, say a, b, c, and d. Suppose that the order in which these objects are listed represents the outcome of an experiment. Let the events A and B be defined as follows: $A = \{a \text{ is in the first position}\}$; $B = \{b \text{ is in the second position}\}$.

   (a) List all elements of the sample space.

   (b) List all elements of the events $A \cap B$ and $A \cup B$.

9. A lot contains items weighing $5, 10, 15, \ldots, 50$ pounds. Assume that at least two items of each weight are found in the lot. Two items are chosen from the lot. Let X denote the weight of the first item chosen and Y the weight of the second item. Thus the pair of numbers (X, Y) represents a single outcome of the experiment. Using the XY-plane, indicate the sample space and the following events.

   (a) $\{X = Y\}$

   (b) $\{Y > X\}$

   (c) The second item is twice as heavy as the first item.

   (d) The first item weighs 10 pounds less than the second item.

   (e) The average weight of the two items is less than 30 pounds.

10. During a 24-hour period, at some time X, a switch is put into "ON" position. Subsequently, at some future time Y (still during that same 24-hour period) the switch is put into the "OFF" position. Assume that X and Y are measured in hours on the time axis with the beginning of the time period as the origin. The outcome of the experiment consists of the pair of numbers (X, Y).

    (a) Describe the sample space.

    (a) Describe and sketch in the XY-plane the following events.

    (i) The circuit is on for one hour or less.

    (ii) The circuit is on at time z where z is some instant during the given 24-hour period.

    (iii) The circuit is turned on before time t_1 and turned off after time t_2 (where again $t_1 < t_2$ are two time instants during the specified period)

    (iv) The circuit is on twice as long as it is off.

11. Let A, B, and C be three events associated with an experiment. Express the following verbal statements in set notation.

    (a) At least one of the events occurs.

    (b) Exactly one of the events occurs.

(c) Exactly two of the events occur.

(d) Not more than two of the events occur simultaneously.

12. Prove that if A, B, and C are any three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ -P(B \cap C) + P(A \cap B \cap C).$$

13. (a) Show that for any two events, A_1 and A_2 we have $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$.

    (b) Show that for any n events $A_1, \ldots, A_n$, we have

$$P(A_1 \cup \cdots \cup A_n) \leq P(A_1) + \cdots + P(A_n).$$

[Hint: Use mathematical induction. The result stated in (b) is called **Boole's inequality**.]

14. Theorem 1.3 deals with the probability that at least one of the two events A or B occurs. The following statement deals with the probability that exactly one of the events A or B occurs.

    Show that $[P(A \cap \bar{B}) \cup (B \cap \bar{A})] = P(A) + P(B) - 2P(A \cap B)$.

15. A certain type of electric motor fails either by seizure of the bearings, or by burning out of the electric windings, or by wearing out of the brushes. Suppose that seizure is twice as likely as burning out, which is four times as likely as brush wearout. What is the probability that failure will be by each of these three mechanisms?

16. Suppose that A and B are events for which $P(A) = x, P(B) = y$, and $P(A \cap B) = z$. Express each of the following probabilities in terms of x, y, and z.

    (a) $P(\bar{A} \cup \bar{B})$

    (b) $P(\bar{A} \cap B)$

    (c) $P(\bar{A} \cup B)$

    (d) $P(\bar{A} \cap \bar{B})$

17. Suppose that A, B, and C are events such that $P(A) = P(B) = P(C) = \frac{1}{4}, P(A \cap B) = P(C \cap B) = 0$, and $P(A \cap C) = \frac{1}{8}$. Evaluate the probability that at least one of the events A, B, or C occurs.

18. An installation consists of two boilers and one engine. Let the event A be that the engine is in good condition, while the events $B_k (k = 1, 2)$ are the events that the kth boiler is in good condition. The event C is that the installation can operate. If the installation is operative whenever the engine and at least one boiler function, express C and $\bar{C}$ in terms of A and the B_i's.

19. A mechanism has two types of parts, say I and II. Suppose that there are two of type I and three of type II. Define the events $A_k, k = 1, 2$, and $B_j, = 1, 2, 3$ as follows: A_k : the kth unit of type I is functioning properly; B_j : the jth unit of type II is functioning properly. Finally, let C represent the event: the mechanism functions. Given that the mechanism functions if at least one unit of type I and at least two units of type II function, express the event C in terms of the A_k's and B_j's.

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